PRELIMINARY EXAM IN ANALYSIS JUNE 2016

INSTRUCTIONS:

(1) There are **three** parts to this exam: I (measure theory), II (functional analysis), and III (complex analysis). Do **three** problems from each part.

(2) In each problem, full credit requires proving that your answer is correct. You may quote and use theorems and formulas. But if a problem asks you to state or prove a theorem or a formula, you need to provide the full details.

Part I. Measure Theory

Do **three** of the following five problems.

(1) (a) State the three convergence theorems for Lebesgue integrals: (1) the monotone convergence theorem; (2) Fatou's lemma; (3) the dominated convergence theorem.

(b) Use the monotone convergence theorem to prove Fatou's lemma.

(2) Suppose that $f : X \to \mathbb{R}$ is integrable on a measure space $(X, \mathscr{F}, \mu)$. Show that for any $\epsilon > 0$ there is a strictly positive δ such that

$$\int_C |f| \, d\mu \leq \epsilon$$

for all measurable sets $C \in \mathscr{F}$ such that $\mu(C) \leq \delta$.

(3) Let $(X, \mathscr{F}, \mu)$ be a measure space and $0 < p < \infty$. Suppose that $\{f_n\}$ is a sequence of L^p-integrable functions such that $f_n \to f$ almost everywhere and f is also L^p-integrable. If furthermore,

$$\int_X |f_n|^p \, d\mu \to \int_X |f|^p \, d\mu,$$

show that

$$\lim_{n\to\infty} \int_X |f_n - f|^p \, d\mu = 0.$$

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be an L^p-integrable function on $\mathbb{R}$ (with respect to the Lebesgue measure and $p \geq 1$). Define for $h > 0$,

$$f_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t) \, dt.$$

Show that $\|f_h\|_p \leq \|f\|_p$ and

$$\lim_{h\to 0} \int_{\mathbb{R}} |f_h(x) - f(x)|^p \, dx = 0.$$

(5) Let $(X, \mathscr{F}, \mu)$ be a measure space and f a nonnegative measurable function. Show that for all $p > 0$,

$$\int_X f^p \, d\mu = p \int_0^\infty t^{p-1} \mu \{f \geq t\} \, dt.$$

Part II. Functional Analysis

Do **three** of the following five problems.

(1) Let $\mathcal{F} : L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}, dx)$ denote the Fourier transform on $\mathbb{R}$. Let $f \in C[0,1]$ and let

$$F(\xi) = \mathcal{F}(f \mathbf{1}_{[0,1]})(\xi) = \int_0^1 e^{-ix\xi} f(x) dx.$$

(a) Show that $F(\xi)$ is a bounded analytic function of the real variable $\xi \in \mathbb{R}$ and find its Taylor expansion centered at $\xi = 0$. What is the radius of convergence?

(b) Does there exist $f \in C[0,1]$ such that

$$\int_0^1 x^n f(x) dx = \begin{cases} 1, & n = 1; \\ 0, & n \geq 2\,? \end{cases}$$

(2) Let H be a separable Hilbert space and let $T : H \to H$ be a compact self-adjoint linear operator. Prove that either $||T||$ or $-||T||$ is an eigenvalue of T. Also, define 'compact', 'self-adjoint', $||T||$ and 'eigenvalue'.

(3) Let H be an infinite dimensional separable Hilbert space. Let $T : H \to H$ be a compact injective operator. Can T be surjective? Prove that your answer is correct.

(4) Prove or disprove:

(a) There is a bounded linear function $\Lambda : L^\infty([-1,1]) \to \mathbb{R}$ such that $\Lambda u = u(0)$ for bounded functions continuous at 0 .

(b) There is a bounded linear function $\Lambda : L^\infty([-1,1]) \to \mathbb{R}$ such that $\Lambda u = u'(0)$ for bounded functions which are differentiable at 0 .

(5) Let $1 < p < \infty$ and let $f_n \in L^p([0,1], dx)$, $||f_n||_p \leq 1$ and assume $f_n \to 0$ almost everywhere.

(a) Show that $f_n \to 0$ weakly in L^p. (Hint: Egorov).

(b) Is this always the case if $p = 1$? If so, prove it; if not, give a counterexample.

Part III. Complex Analysis

Do **three** of the following five problems.

(1) (a) Show that all bijective analytic maps $f : \mathbb{C} \to \mathbb{C}$ are of the form $f(z) = az + b$ for complex numbers a, b with $a \neq 0$.

(b) Classify all **injective** analytic maps $f : \mathbb{C} \to \mathbb{C}$.

(2) (a) Show that for $w \in \mathbb{C}$ with $|w| > 1$,

$$\int_0^{2\pi} \frac{w - e^{i\theta}}{w - e^{-i\theta}} d\theta = 2\pi \left(1 - \frac{1}{w^2}\right).$$

(b) Compute the same integral as in (a) for $|w| < 1$.

(3) Fix $a \in \mathbb{C}$ with $|a| < 1$ and a positive integer n. How many roots does the equation

$$z^n = ae^{-z-1}$$

have in the unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$? What are their multiplicities?

(4) Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disc. Suppose that $f : \mathbb{D} \to \mathbb{C}$ is an analytic map satisfying $|f(z)| < R$ for some $R > 0$. Show that

$$\left| \frac{f(z) - f(0)}{R^2 - \overline{f(0)} f(z)} \right| \leq \frac{|z|}{R}.$$

(5) Let D be a bounded domain in $\mathbb{C}$ and let φ be a bounded real-valued function on ∂D. Let $u : D \to \mathbb{R}$ be the Perron solution of the corresponding Dirichlet problem, namely

$$u(z) = \sup\{v(z) \mid v \in C^0(\overline{D}),\ v \text{ is subharmonic on } D,\ v \leq \varphi \text{ on } \partial D\},$$

which is harmonic in D (you do not need to prove this). Assume:

(i) $0 \in \partial D$ and $\{|z-1| \leq 1\} \cap \overline{D} = \{0\}$.

(ii) $\varphi(0) = 0$ and φ is continuous at 0.

Show that:

(a) $z \mapsto \log|z-1|$ is a harmonic function on D.

(b) $u(z) \to 0$ as $z \to 0$.

(Hint: for (b) show that given $\epsilon > 0$, there exists A sufficiently large such that $-\epsilon - A\log|z-1| \leq u(z) \leq \epsilon + A\log|z-1|$.)